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In-plane complex potentials for a special class of materials with degenerate piezoelectric properties

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Abstract

In this paper, complex potentials for the solution of two-dimensional, in-plane, linear piezoelectric boundary value problems are presented. These potentials are only valid for a special set of piezoelectric properties that have been identified as being useful in nonlinear ferroelectric constitutive laws. In contrast to more general solution procedures like the Stroh or Lekhnitskii formalisms, the complex potentials derived here are dependent on explicit, closed-form combinations of the piezoelectric material properties. Under either plane strain or plane stress conditions, three complex potentials are required to determine the full set of electrical and mechanical field quantities. The components of stress, strain, displacement, electric field, electric displacement, and electric potential will all be given in terms of these three potentials. To demonstrate the solution to a boundary value problem with these potentials, the asymptotic fields near a crack tip in these materials are presented in closed form.

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1. Introduction

Over the past few decades, the analysis of linear piezoelectric boundary value problems has become relatively well-developed (Barnett and Lothe, 1975; Deeg, 1980; Sosa, 1991, 1992; Suo et al., 1992; Pak, 1992; Park and Sun, 1995). These works are essentially extensions of the anisotropic elasticity formalisms of Lekhnitskii (1950) or Eshelby et al. (1953) and Stroh (1958). More recently, efforts have been made to establish non-linear phenomenological constitutive laws for ferroelectric materials (Kamlah, 2001; Landis, 2002; McMeeking and Landis, 2002). These types of constitutive laws have potential use for the analysis of actuator and sensor devices and for the study of the electromechanical fracture behavior of ferroelectrics. An interesting feature of ferroelectric ceramics that must be incorporated into these non-linear constitutive laws is that the elastic, dielectric and most importantly the piezoelectric properties of the material can change as the remanent polarization and strain in the material evolve. This is in contrast to plasticity in

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polycrystalline metals, where the elastic properties of the material are essentially independent of the plastic deformation.

A considerable simplification to these non-linear constitutive laws is made if the linear properties of the material are assumed to take the following forms, Landis (2002),

$$s_{ijkl}^E = \frac{1+v}{2E} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{v}{E} \delta_{ij}\delta_{kl} \quad (1.1)$$

$$\kappa_{ij}^e = \kappa\delta_{ij} \quad (1.2)$$

$$d_{kij} = \frac{P^r}{P_0} \left[d_{33}m_k m_i m_j + d_{31}m_k \lambda_{ij} + \frac{d_{15}}{2} (m_i \lambda_{jk} + m_j \lambda_{ik}) \right] \quad (1.3)$$

where δ_{ij} is the Kronecker delta, the Cartesian components of the remanent polarization vector are P_i^r , its magnitude is $P^r = \sqrt{P_i^r P_i^r}$, the components of its direction are $m_i = P_i^r / P^r$ and the components of the transversely isotropic second rank tensor λ are $\lambda_{ij} = \delta_{ij} - m_i m_j$. The components of the elastic compliance measured at constant electric field are s_{ijkl}^E , the dielectric permittivities at constant stress are κ_{ij}^e , and the piezoelectric coefficients are d_{kij} . Finally, E and v are the Young's modulus and Poisson's ratio of the material at constant electric field, κ is the dielectric permittivity at constant stress, and d_{33} , d_{31} and d_{15} are the piezoelectric coefficients in standard Voight notation. Note that the elastic compliance and dielectric permittivity are isotropic tensors, and the piezoelectric tensor is transversely isotropic about the remanent polarization direction \mathbf{m} .

The reason why the forms for the linear properties given in Eqs. (1.1)–(1.3) simplify the non-linear constitutive theories for ferroelectrics is that in these theories, derivatives of the linear properties with respect to the remanent polarization and remanent strain components are required, Landis (2002). Inspection of (1.1)–(1.3) yields the fact that none of the properties depends on the remanent strain components and hence all derivatives with respect to the remanent strains are zero. Furthermore, the elastic compliance and dielectric permittivity do not depend on the remanent polarization, so the derivatives of these tensors with respect to the P_i^r are zero as well. Finally, the piezoelectric properties d_{kij} do depend on P_i^r , and this is a physical requirement for ferroelectrics. This feature manifests itself in the fact that unpoled ferroelectrics are not piezoelectric, but poled ferroelectrics exhibit piezoelectricity. However, one final simplification can be made to the piezoelectric properties that further simplifies the non-linear ferroelectric constitutive laws. This simplification is obtained by requiring that

$$\frac{\partial^2 d_{kij}}{\partial P_m^r \partial P_n^r} = 0 \rightarrow d_{15} = d_{33} - d_{31} \quad (1.4)$$

While mathematical simplicity is a noble goal in any model of a physical system, such desire for simplicity is always superseded by the need for physical authenticity. Hence, the question as to whether $d_{15} = d_{33} - d_{31}$ is a reasonable approximation must be addressed. The answer to this question is, in fact, yes for poled polycrystalline ferroelectric ceramics. For example, the properties for poled barium titanate are $d_{33} = 5.73 \times 10^{-11}$ C/N, $d_{31} = -2.37 \times 10^{-11}$ C/N and $d_{15} = 8.10 \times 10^{-11}$ C/N as reported by Berlincourt and Jaffe (1958). Also, the properties reported by Deeg (1980) and Pak (1992) for PZT-5H poled ceramic are $d_{33} = 3.15 \times 10^{-10}$ C/N, $d_{31} = -1.28 \times 10^{-10}$ C/N and $d_{15} = 4.82 \times 10^{-10}$ C/N. Hence, it is reasonable to make the assumption that $d_{15} = d_{33} - d_{31}$ for poled ceramics. Note that it is not advisable to make this assumption for single crystal ferroelectrics.

Given the properties in Eqs. (1.1)–(1.3) along with the simplification that $d_{15} = d_{33} - d_{31}$, it is useful to develop a linear piezoelectricity theory for in-plane electromechanical loading on such a material. The initial procedure to solve this problem is to apply one of the well-established formalisms for anisotropic linear piezoelectricity. However, the properties in (1.1)–(1.3) with $d_{15} = d_{33} - d_{31}$ are mathematically

degenerate, and therefore the standard Stroh or Lekhnitskii procedures must be modified to account for this degeneracy. In the following, an approach similar to those of Stroh and Lekhnitskii is used to determine complex potentials that will solve in-plane piezoelectricity problems with the material properties described above. One of the benefits of these assumed forms of the piezoelectric properties is that the solutions to the in-plane problems can be given in closed-form without the need for the numerical solution of an eigenvalue problem. Finally, looking forward to the fact that non-linear small scale switching analyses will eventually be performed with the constitutive law mentioned above, solutions for the asymptotic crack tip fields for both conducting and impermeable boundary conditions will be obtained with these complex potentials.

The remainder of this paper is organized as follows. Section 2 outlines the equations governing a two-dimensional, in-plane, linear piezoelectric boundary value problem. Section 3 then describes the solution procedures for these equations in both plane strain and plane stress. Section 4 applies the complex potentials derived in Section 3 to the solution of the asymptotic crack tip solution. Finally, a short discussion of the results is given in Section 5.

2. Governing equations

In this section the equations governing a linear piezoelectric boundary value problem will be presented. Throughout this paper it is assumed that the material is poled along the x_3 direction, which forms an angle of β with the y -direction as in Fig. 1. The mechanical field equations will be presented first, followed by the electrical equations. In the absence of body forces, mechanical equilibrium in the volume of the body is given as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad (2.1)$$

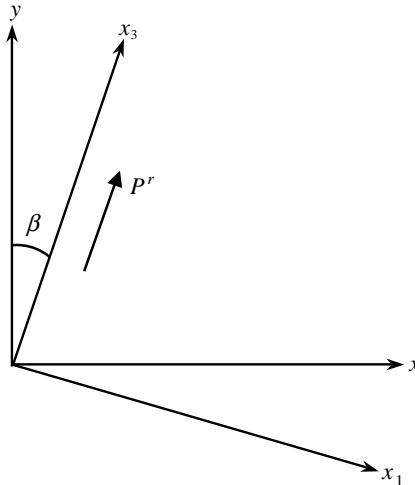


Fig. 1. The coordinate systems for the analysis boundary value problems where the remanent polarization direction lies at an arbitrary angle β from the y -axis. Note that the x_3 -axis is parallel to the remanent polarization direction by convention. Furthermore, the indices used for Voigt notation are base on this 1-2-3 system.

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (2.2)$$

where σ_{xx} , σ_{yy} and σ_{xy} are Cartesian components of the Cauchy stress tensor. On the surface of the body the stresses must be in equilibrium with the surface tractions as

$$t_x = \sigma_{xx}n_x + \sigma_{xy}n_y \quad (2.3)$$

$$t_y = \sigma_{xy}n_x + \sigma_{yy}n_y \quad (2.4)$$

where t_x , t_y , n_x and n_y are the components of the traction vector and the outward unit vector normal to the surface. The strain–displacement relationships are

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} \quad (2.5)$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} \quad (2.6)$$

$$\epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \quad (2.7)$$

where ϵ_{xx} , ϵ_{yy} and ϵ_{xy} are the components of the infinitesimal strain tensor, and u_x and u_y are the components of the displacement vector.

The electrical equations are as follows. In the absence of a free charge density distribution, Guass' law in the volume of the material dictates that

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \quad (2.8)$$

where D_x and D_y are the components of the electric displacement vector. On the surface of the body,

$$\omega = -D_x n_x - D_y n_y \quad (2.9)$$

where ω is the surface free charge density. Finally, the electric field components, E_x and E_y , can be derived from the electric potential ϕ as

$$E_x = -\frac{\partial \phi}{\partial x} \quad (2.10)$$

$$E_y = -\frac{\partial \phi}{\partial y} \quad (2.11)$$

Eqs. (2.1)–(2.11) represent eight governing equations for 13 independent field quantities. Note that (2.3), (2.4) and (2.9) are surface or boundary equations. The remaining five equations required to close the loop on a given boundary value problem are the constitutive equations for the piezoelectric material. As noted previously, it is assumed that the material is poled in the x_3 -direction. Furthermore, the piezoelectric properties take on the special forms described in Section 1. Specifically, the constitutive law can be written as

$$\epsilon_{xx} = \frac{1}{E} \sigma_{xx} - \frac{v}{E} \sigma_{yy} - \frac{v}{E} \sigma_{zz} + (d_{33} \sin \beta) E_x + (d_{31} \cos \beta) E_y \quad (2.12)$$

$$\epsilon_{yy} = -\frac{v}{E} \sigma_{xx} + \frac{1}{E} \sigma_{yy} - \frac{v}{E} \sigma_{zz} + (d_{31} \sin \beta) E_x + (d_{33} \cos \beta) E_y \quad (2.13)$$

$$\epsilon_{zz} = -\frac{v}{E} \sigma_{xx} - \frac{v}{E} \sigma_{yy} - \frac{1}{E} \sigma_{zz} + (d_{31} \sin \beta) E_x + (d_{31} \cos \beta) E_y \quad (2.14)$$

$$\varepsilon_{xy} = \frac{1+v}{E} \sigma_{xy} + \frac{d_{33} - d_{31}}{2} (E_x \cos \beta + E_y \sin \beta) \quad (2.15)$$

$$D_x = (d_{33} \sin \beta) \sigma_{xx} + (d_{31} \sin \beta) \sigma_{yy} + (d_{31} \sin \beta) \sigma_{zz} + [(d_{33} - d_{31}) \cos \beta] \sigma_{xy} + \kappa E_x \quad (2.16)$$

$$D_y = (d_{31} \cos \beta) \sigma_{xx} + (d_{33} \cos \beta) \sigma_{yy} + (d_{31} \cos \beta) \sigma_{zz} + [(d_{33} - d_{31}) \sin \beta] \sigma_{xy} + \kappa E_y \quad (2.17)$$

Here E and v are Young's modulus and Poisson's ratio of the material measured at constant electric field. Note that an E without a subscript is used to denote Young's modulus, and an E with a subscript is used to denote an electric field component. The out of plane axial stress is denoted as σ_{zz} . The piezoelectric coefficients are d_{33} and d_{31} . Here, the 1, 2, 3 notation follows standard Voight notation for piezoelectric materials with the three directions aligned with the remanent polarization. Also note that in Voight notation, this form of the material properties assumes that $d_{15} = d_{33} - d_{31}$. Finally, κ is the dielectric permittivity of the material measured at constant stress. Again, we emphasize that Eqs. (2.12)–(2.17) are not the most general form for a poled ceramic, but rather a very specific special form of the linear constitutive behavior which is useful within nonlinear material laws for ferroelectrics described in Section 1.

Lastly, a few caveats should be mentioned when applying Eqs. (2.1)–(2.17) to poled ferroelectrics. First, these equations are valid for a material sample with a uniform distribution of remanent polarization and therefore a uniform distribution of piezoelectric properties. Furthermore, the strain and electric displacement components appearing in these equations are actually *changes* from the zero stress and zero electric field remanent configuration. This also implies that when there is no stresses or electric fields applied to the sample, there will be a surface free charge density on any surface with a component of its unit normal parallel to the remanent polarization direction. Hence, the surface free charge density ω appearing in Eq. (2.9) is actually the level of free charge above the reference level of $\omega_0 = -P_i^r n_i$ which is required to equilibrate the initial remanent polarization. Finally, the constitutive equations are only valid in the absence of domain switching. In other words the remanent polarization and remanent strain must remain fixed at all points in the body.

3. Solution procedure

3.1. Plane strain

For the plane strain problem, the axial strain normal to the x – y plane is set to zero, and the out of plane axial stress can be solved as

$$\sigma_{zz} = v(\sigma_{xx} + \sigma_{yy}) - Ed_{31}(E_x \sin \beta + E_y \cos \beta) \quad (3.1)$$

Now the Airy's stress function χ is introduced such that the equilibrium equations, (2.1) and (2.2), are satisfied if

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2} \quad (3.2)$$

$$\sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \quad (3.3)$$

$$\sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \quad (3.4)$$

Furthermore, Eqs. (2.5)–(2.7) can be combined into a compatibility equation as

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \quad (3.5)$$

Now Eqs. (3.2)–(3.4) and (2.6) can be substituted into (3.1) and the constitutive Eqs. (2.12)–(2.17). Then, the constitutive equations for the strains and electric displacements can be substituted into the compatibility equation (3.5) and Gauss' law (2.8). Eqs. (3.5) and (2.8) then result in two governing partial differential equations for the Airy's stress function and the electric potential. The final simplified forms for these equations are as follows:

$$\frac{1-v^2}{E} \nabla^4 \chi - d_{31}(1+v) \left(\sin \beta \frac{\partial}{\partial x} + \cos \beta \frac{\partial}{\partial y} \right) \nabla^2 \phi = 0 \quad (3.6)$$

$$d_{31}(1+v) \left(\sin \beta \frac{\partial}{\partial x} + \cos \beta \frac{\partial}{\partial y} \right) \nabla^2 \chi - \kappa \nabla^2 \phi + E d_{31}^2 \left(\sin \beta \frac{\partial}{\partial x} + \cos \beta \frac{\partial}{\partial y} \right)^2 \phi = 0 \quad (3.7)$$

where ∇^2 is the two-dimensional Laplacian operator, and $\nabla^4 = \nabla^2 \nabla^2$ is the biharmonic operator. The general solution to these equations can be found by taking χ and ϕ to be functions of a complex variable $z_p = x + py$ (p is complex) in the following ways

$$\chi = a_\chi f(z_p) \quad (3.8)$$

$$\phi = a_\phi f'(z_p) \quad (3.9)$$

where $f'(z_p) = df/dz_p$. Using the relationships

$$\frac{\partial}{\partial x} [f(z_p)] = f'(z_p) \quad (3.10)$$

$$\frac{\partial}{\partial y} [f(z_p)] = p f'(z_p) \quad (3.11)$$

Eqs. (3.8) and (3.9) can be substituted into (3.6) and (3.7). This results in an eigenvalue problem with p as the eigenvalue and (a_χ, a_ϕ) as the associated eigenvector. The solutions for the eigenvalues are

$$p_{\pm 1, \pm 2, \pm 3} = \pm i, \pm i, \frac{k_e \sin \beta \cos \beta \pm i \sqrt{1 - k_e}}{1 - k_e \cos^2 \beta} \quad (3.12)$$

where $i = \sqrt{-1}$ and the solutions $\pm i$ have been explicitly repeated to indicate that a double root exists. Furthermore, the plane strain electromechanical coupling coefficient k_e is

$$k_e = \frac{2E d_{31}^2}{\kappa(1-v)} \quad (3.13)$$

Note that the system of Eqs. (3.6) and (3.7) remains elliptic if $k_e < 1$. Furthermore, this condition is automatically satisfied if the material is stable, i.e. if any set of applied stresses and electric fields leads to positive stored energy in the material. Proof of this fact is readily obtained by noting that the eigenvalues of the material matrix relating the subset $(\varepsilon_{xx}, \varepsilon_{zz}, D_y)$ to $(\sigma_{xx}, \sigma_{zz}, E_y)$ must all be positive for a stable material.

The third set of eigenvalues will be renamed such that

$$p_e = \frac{k_e \sin \beta \cos \beta + i \sqrt{1 - k_e}}{1 - k_e \cos^2 \beta} \quad \text{and} \quad \bar{p}_e = \frac{k_e \sin \beta \cos \beta - i \sqrt{1 - k_e}}{1 - k_e \cos^2 \beta}. \quad (3.14)$$

Throughout this work, an overbar will always represent the complex conjugate of the variable below. Note that double roots exist in Eq. (3.12), and these identical eigenvalues do not have distinct eigenvectors associated with them. This fact implies that a second solution of a form different from Eqs. (3.8) and (3.9), corresponding to the second set of eigenvalues at $\pm i$, must be determined. This second solution takes the form

$$\chi = a_\chi \bar{z}f(z) + b_\chi zg(\bar{z}) \quad (3.15)$$

$$\phi = a_\phi f(z) + b_\phi g(\bar{z}) \quad (3.16)$$

where $z = x + iy$ and $\bar{z} = x - iy$. Finally, by applying the fact that both χ and ϕ are real, it can be shown that the general solution to Eqs. (3.4) and (3.5) takes the form

$$\chi = \operatorname{Re}[F(z)] + \operatorname{Re}[\bar{z}G(z)] + \operatorname{Re}[H(z_e)] \quad (3.17)$$

$$\phi = -\frac{4(1+v)}{Ed_{31}} \operatorname{Re}[(\sin \beta - i \cos \beta)G(z)] + \frac{1-v}{Ed_{31}} \operatorname{Re}\left[\frac{p_e^2 + 1}{\sin \beta + p_e \cos \beta} H'(z_e)\right] \quad (3.18)$$

Note that F and G are analytic functions of the variable $z = x + iy$, and H is an analytic function of the variable $z_e = x + p_e y$.

Application of Eqs. (3.2)–(3.4) and (2.10) and (2.11) allows for the determination of the stress and electric field components as

$$\sigma_{xx} = -\operatorname{Re}F'' + \operatorname{Re}(2G' - \bar{z}G'') + \operatorname{Re}(p_e^2 H'') \quad (3.19)$$

$$\sigma_{yy} = \operatorname{Re}F'' + \operatorname{Re}(2G' + \bar{z}G'') + \operatorname{Re}H'' \quad (3.20)$$

$$\sigma_{xy} = \operatorname{Im}F'' + \operatorname{Im}(\bar{z}G'') - \operatorname{Re}(p_e H'') \quad (3.21)$$

$$E_x = \frac{4(1+v)}{Ed_{31}} \operatorname{Re}[(\sin \beta - i \cos \beta)G'] - \frac{1-v}{Ed_{31}} \operatorname{Re}\left[\frac{p_e^2 + 1}{\sin \beta + p_e \cos \beta} H''\right] \quad (3.22)$$

$$E_y = -\frac{4(1+v)}{Ed_{31}} \operatorname{Im}[(\sin \beta - i \cos \beta)G'] - \frac{1-v}{Ed_{31}} \operatorname{Re}\left[\frac{p_e^3 + p_e}{\sin \beta + p_e \cos \beta} H''\right] \quad (3.23)$$

The determination of the displacements requires the exploitation of the Cauchy–Riemann conditions, the constitutive equations, and strain–displacement relations. It can be shown that to within a rigid body motion

$$\begin{aligned} \frac{E}{1+v} u_x &= \operatorname{Re} \left\{ -F' + \left[7 + 4 \frac{d_{33} - d_{31}}{d_{31}} \sin \beta (\sin \beta - i \cos \beta) \right] G - \bar{z}G' \right. \\ &\quad \left. - \left[1 + \frac{1-v}{1+v} \frac{d_{33} - d_{31}}{d_{31}} \frac{\sin \beta}{\sin \beta + p_e \cos \beta} \right] H' \right\} \end{aligned} \quad (3.24)$$

$$\begin{aligned} \frac{E}{1+v} u_y &= \operatorname{Im} \left\{ F' + \left[7 + 4 \frac{d_{33} - d_{31}}{d_{31}} \cos \beta (\cos \beta + i \sin \beta) \right] G + \bar{z}G' \right\} \\ &\quad - \operatorname{Re} \left\{ \left[p_e + \frac{1-v}{1+v} \frac{d_{33} - d_{31}}{d_{31}} \frac{(p_e^2 + 1) \cos \beta}{\sin \beta + p_e \cos \beta} \right] H' \right\}. \end{aligned} \quad (3.25)$$

Finally, the strain components can either be determined from the stresses and electric fields through the constitutive law (2.12)–(2.15), or from the displacements through Eqs. (2.5)–(2.7). Also, the electric displacement components can be determined from the stress and electric field components through (2.16) and (2.17).

3.2. Plane stress

For the plane stress problem $\sigma_{zz} = 0$. Eqs. (3.2)–(3.5) are still valid and the procedure for obtaining the governing equations for χ and ϕ is identical to that used in the plane strain case. The resulting forms of the compatibility equation and Gauss' law are

$$\frac{1}{E} \nabla^4 \chi - d_{31} \left(\sin \beta \frac{\partial}{\partial x} + \cos \beta \frac{\partial}{\partial y} \right) \nabla^2 \phi = 0 \quad (3.26)$$

$$d_{31} \left(\sin \beta \frac{\partial}{\partial x} + \cos \beta \frac{\partial}{\partial y} \right) \nabla^2 \chi - \kappa \nabla^2 \phi = 0 \quad (3.27)$$

Following the solution procedures described previously, three sets of eigenvalues analogous to those found in Eq. (3.12) exist. However, for the eigenvalues associated with the repeated roots at $\pm i$ there exist two distinct sets of eigenvectors. Therefore, the solution to Eqs. (3.26) and (3.27) has the form

$$\chi = \operatorname{Re}[P(z)] + \operatorname{Re}[Q(z_\sigma)] \quad (3.28)$$

$$\phi = -\operatorname{Im}[S'(z)] + \frac{d_{31}}{\kappa} \operatorname{Re}[(\sin \beta + p_\sigma \cos \beta) Q'(z_\sigma)] \quad (3.29)$$

where

$$z_\sigma = x + p_\sigma y, \quad p_\sigma = \frac{k_\sigma \sin \beta \cos \beta + i\sqrt{1 - k_\sigma^2}}{1 - k_\sigma \cos^2 \beta}, \quad \text{and} \quad k_\sigma = \frac{Ed_{31}^2}{\kappa}. \quad (3.30)$$

Here, the governing equations remain elliptic if $k_\sigma < 1$. As for the case of plane strain, this condition is automatically satisfied if the material is stable. Proof of this fact can be obtained by noting that the eigenvalues of the material matrix relating (ε_{xx}, D_y) to (σ_{xx}, E_y) must all be positive for a stable material.

Again, note that the potentials P and S are analytic functions of z and Q is an analytic function of z_σ . For plane stress, the stress and electric field components are given as

$$\sigma_{xx} = -\operatorname{Re}P'' + \operatorname{Re}(p_\sigma^2 Q'') \quad (3.31)$$

$$\sigma_{yy} = \operatorname{Re}P'' + \operatorname{Re}Q'' \quad (3.32)$$

$$\sigma_{xy} = \operatorname{Im}P'' - \operatorname{Re}(p_\sigma Q'') \quad (3.33)$$

$$E_x = \operatorname{Im}S'' - \frac{d_{31}}{\kappa} \operatorname{Re}[(\sin \beta + p_\sigma \cos \beta) Q''] \quad (3.34)$$

$$E_y = \operatorname{Re}S'' - \frac{d_{31}}{\kappa} \operatorname{Re}[p_\sigma(\sin \beta + p_\sigma \cos \beta) Q''] \quad (3.35)$$

Finally, to within a rigid body motion, the displacements are given as

$$\begin{aligned} \frac{E}{1+v} u_x &= -\operatorname{Re}P' + \frac{1}{1+v} \operatorname{Re} \left\{ \left[p_\sigma^2 - v - k_\sigma(\sin \beta + p_\sigma \cos \beta) \left(p_\sigma \cos \beta + \frac{d_{33}}{d_{31}} \sin \beta \right) \right] Q' \right\} \\ &+ \frac{Ed_{31}}{1+v} \operatorname{Re} \left[\left(\cos \beta - i \frac{d_{33}}{d_{31}} \sin \beta \right) S' \right] \end{aligned} \quad (3.36)$$

$$\begin{aligned} \frac{E}{1+v}u_y &= \operatorname{Im} P' + \frac{1}{1+v} \operatorname{Re} \left\{ \left[\frac{1}{p_\sigma} - p_\sigma v - \frac{k_\sigma}{p_\sigma} (\sin \beta + p_\sigma \cos \beta) \left(\frac{d_{33}}{d_{31}} p_\sigma \cos \beta + \sin \beta \right) \right] Q' \right\} \\ &\quad + \frac{Ed_{31}}{1+v} \operatorname{Im} \left[\left(\frac{d_{33}}{d_{31}} \cos \beta - i \sin \beta \right) S' \right] \end{aligned} \quad (3.37)$$

3.3. Discussion

In this section governing equations for the Airy's stress function χ and the electric potential ϕ have been solved using complex variable methods. This is in contrast with the Stroh approach, which solves governing equations for the displacements u_x and u_y and the electric potential, or the approach of Sosa (1991) who solved equations for Airy's stress function and an induction potential ψ that was used to derive electric displacement components. An approach similar to that of Sosa, but along the lines of Lekhnitskii where the single Airy's stress function χ is replaced by two components of its vectorial counterpart can also be used. Finally, a fourth approach using displacements and the induction potential could be applied to the problem as well. Obviously, these seemingly different methods are intimately related to one another since they each solve the same problem. The reasons why one approach is or should be chosen over another involve the simplicity with which the constitutive law can be represented, and the types of boundary conditions that are presented in a given problem. For example, for boundary value problems where only tractions and electric potentials are applied to the surface, the approach using χ and ϕ offers a small advantage over the others when analytical solutions are possible. However, since the eigenvalues and eigenvectors are in many cases determined numerically for general forms of the piezoelectric properties, we emphasize that this advantage is slight. The primary reason for using χ and ϕ in this work is due to the specific form of the linear piezoelectric properties, i.e. Eqs. (2.12)–(2.17).

Finally, note that the stresses for the plane stress case in Eqs. (3.31)–(3.33) depend on only two of the three complex potentials. This fact implies that for problems where the mechanical boundary conditions *only* contain specified *tractions*, then the two potentials $P(z)$ and $Q(z_\sigma)$ are not dependent on the electrical boundary conditions specified in the problem. This feature of the plane stress solutions will be illustrated in the next section in Tables 3 and 4, where the coefficients of $P(z)$ and $Q(z_\sigma)$ are shown to be independent of the electrical crack face boundary conditions.

4. Asymptotic crack tip fields

Due to the inherent brittleness of piezoelectric ceramics, the fracture behavior of these materials has been the topic of considerable of study, (Sosa, 1991; Suo et al., 1992; Pak, 1992; Dunn, 1994; Park and Sun, 1995; McMeeking, 2001 among others). In this section, the complex potentials derived in Section 3 will be used to determine the electrical and mechanical fields near the tip of a traction free crack in a linear piezoelectric material with the properties described in Section 1. The problem will be solved for both electrically conducting and electrically impermeable crack face boundary conditions. Full solutions will be given for a material poled perpendicular to the crack plane and for a material poled parallel to the crack plane. Finally, the Irwin matrix, which relates the energy release rate to the mechanical and electrical intensity factors, will be given for both the conducting and impermeable electrical conditions and arbitrary orientation of the crack with respect to the poling direction.

Fig. 2 is an illustration of the geometry to be analyzed. The traction free boundary conditions imply that

$$t_x = t_y = 0 \rightarrow \sigma_{xy} = \sigma_{yy} = 0 \quad \text{on } \theta = \pm\pi. \quad (4.1)$$

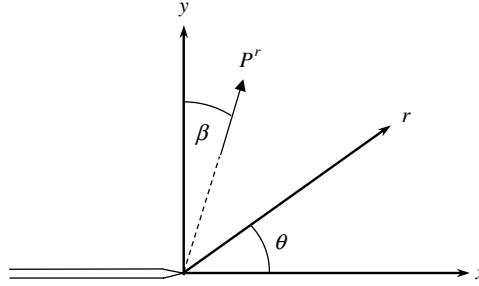


Fig. 2. The coordinate systems for the analysis of the asymptotic crack tip fields in Section 4. The analyses of the complex potentials in Section 4 only give explicit results for $\beta = 0$ and $\beta = \pi/2$. However, the results for the Irwin matrices given in Table 5 are valid for any β through the use of Eqs. (4.78)–(4.83).

Then, the standard stress intensity normalizations imply that

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \quad \text{on } \theta = 0 \quad (4.2)$$

$$\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}} \quad \text{on } \theta = 0 \quad (4.3)$$

where K_I and K_{II} are the mode I and mode II stress intensity factors. For the electrically conducting crack,

$$\phi = 0 \rightarrow E_x = 0 \quad \text{on } \theta = \pm\pi \quad (4.4)$$

$$E_x = \frac{K_E}{\sqrt{2\pi r}} \quad \text{on } \theta = 0 \quad (4.5)$$

Finally, for the electrically impermeable crack,

$$\omega = 0 \rightarrow D_y = 0 \quad \text{on } \theta = \pm\pi \quad (4.6)$$

$$D_y = \frac{K_D}{\sqrt{2\pi r}} \quad \text{on } \theta = 0 \quad (4.7)$$

K_E and K_D are the electric field and electric displacement intensity factors. K_D is also referred to as K_{IV} , Suo et al. (1992). Note that (4.4)–(4.7) will not both be satisfied within a given problem, (4.4) and (4.5) will be satisfied for the conducting crack and (4.6) and (4.7) will hold for the impermeable crack. However, (4.1)–(4.3) are valid for both electrical crack types. Also note that no physical crack is actually impermeable. However, the condition given by (4.6) is valid for the determination of the fields asymptotically close to the crack tip. The consideration of a permeable crack simply affects the level of the intensity factor K_D . For more details on the treatment of permeable cracks see the works of Dunn (1994) and McMeeking (2001).

In all cases we are interested in the dominant terms near the crack tip, i.e. the most singular terms. However, we will enforce the constraint that a finite amount of energy must be stored in any finite volume of material near the crack tip. These considerations ultimately lead to the conclusion that the stress, strain, electric field and electric displacement components each have a $1/\sqrt{r}$ radial dependence.

4.1. Plane strain crack tip fields

Applying the fact that the stresses and electric fields have a $1/\sqrt{r}$ radial dependence, the derivatives of the complex potentials F , G and H can be written as

$$F'' = (a + ib)z^{-1/2} \quad (4.8)$$

$$G' = (c + id)z^{-1/2} \quad (4.9)$$

$$H'' = (e + if)z_e^{-1/2} \quad (4.10)$$

In the following subsections the conditions (4.1)–(4.7) will be applied to determine the real coefficients a, b, c, d, e and f .

4.1.1. $\beta = 0$ crack perpendicular to poling direction, electrically conducting

For $\beta = 0$ it is useful to rewrite Eqs. (3.19)–(3.25). For $\beta = 0$, $p_e = i\alpha_e = i/\sqrt{1 - k_e}$. Therefore, $z_e = x + i\alpha_e y$, where α_e is real. Then, the stresses, electric fields, displacements and electric potential are

$$\sigma_{xx} = -\operatorname{Re} F'' + \operatorname{Re} (2G' - \bar{z}G'') - \alpha_e^2 \operatorname{Re} H'' \quad (4.11)$$

$$\sigma_{yy} = \operatorname{Re} F'' + \operatorname{Re} (2G' + \bar{z}G'') + \operatorname{Re} H'' \quad (4.12)$$

$$\sigma_{xy} = \operatorname{Im} F'' + \operatorname{Im} (\bar{z}G'') + \alpha_e \operatorname{Im} H'' \quad (4.13)$$

$$E_x = \frac{4(1+\nu)}{Ed_{31}} \operatorname{Im} G' + \frac{1-\nu}{Ed_{31}} k_e \alpha_e \operatorname{Im} H'' \quad (4.14)$$

$$E_y = \frac{4(1+\nu)}{Ed_{31}} \operatorname{Re} G' + \frac{1-\nu}{Ed_{31}} k_e \alpha_e^2 \operatorname{Re} H'' \quad (4.15)$$

$$\frac{E}{1+\nu} u_x = -\operatorname{Re} F' + \operatorname{Re} (7G - \bar{z}G') - \operatorname{Re} H' \quad (4.16)$$

$$\frac{E}{1+\nu} u_y = \operatorname{Im} F' + \operatorname{Im} \left[\left(7 + 4 \frac{d_{33} - d_{31}}{d_{31}} \right) G + \bar{z}G' \right] + \left(\alpha_e + \frac{1-\nu}{1+\nu} \frac{d_{33} - d_{31}}{d_{31}} k_e \alpha_e \right) \operatorname{Im} H' \quad (4.17)$$

$$\phi = -\frac{4(1+\nu)}{Ed_{31}} \operatorname{Im} G - \frac{1-\nu}{Ed_{31}} k_e \alpha_e \operatorname{Im} H' \quad (4.18)$$

For the material poled perpendicular to the crack plane, under electrically conducting crack boundary conditions, Eqs. (4.1)–(4.5) imply that the coefficients in Eqs. (4.8)–(4.10) satisfy the following equations.

$$a - \frac{1}{2}c + \alpha_e e = 0 \quad (4.19)$$

$$b + \frac{3}{2}d + f = 0 \quad (4.20)$$

$$a + \frac{3}{2}c + e = \frac{K_I}{\sqrt{2\pi}} \quad (4.21)$$

$$b - \frac{1}{2}d + \alpha_e f = \frac{K_{II}}{\sqrt{2\pi}} \quad (4.22)$$

$$4(1+\nu)c + k_e \alpha_e (1-\nu)e = 0 \quad (4.23)$$

Table 1

The coefficients for the plane strain complex potentials with $\beta = 0$

Plane strain	$\beta = 0$	$k_e = \frac{2Ed_{31}^2}{\kappa(1-v)}$	$\alpha_e = \sqrt{\frac{1}{1-k_e}}$
	$D_E = k_e\alpha_e(1-v) + 2(\alpha_e - 1)(1+v)$		$D_D = k_e(1-v) + 2(\alpha_e - 1)(1+v)$
	Conducting		Impermeable
a	$\frac{k_e\alpha_e(1-v) + 8\alpha_e(1+v)}{4D_E} \frac{K_I}{\sqrt{2\pi}}$		$\frac{k_e(1-v)[d_{33}(1+3\alpha_e) + d_{31}(1-3\alpha_e)] + 16d_{31}\alpha_e(1+v)}{8d_{31}D_D} \frac{K_I}{\sqrt{2\pi}}$ $+ \frac{-k_e(1-v)(1+3\alpha_e)}{8d_{31}D_D} \frac{K_D}{\sqrt{2\pi}}$
b	$\frac{3k_e\alpha_e(1-v) - 8(1+v)}{4D_E} \frac{K_{II}}{\sqrt{2\pi}} - \frac{Ed_{31}(1+3\alpha_e)}{4D_E} \frac{K_E}{\sqrt{2\pi}}$		$\frac{3k_e(1-v) - 8(1+v)}{4D_D} \frac{K_{II}}{\sqrt{2\pi}}$
c	$\frac{k_e\alpha_e(1-v)}{2D_E} \frac{K_I}{\sqrt{2\pi}}$		$\frac{k_e(1-v)[d_{31}(\alpha_e + 1) - d_{33}(\alpha_e - 1)]}{4d_{31}D_D} \frac{K_I}{\sqrt{2\pi}} + \frac{k_e(\alpha_e - 1)(1-v)}{4d_{31}D_D} \frac{K_D}{\sqrt{2\pi}}$
d	$-\frac{k_e\alpha_e(1-v)}{2D_E} \frac{K_{II}}{\sqrt{2\pi}} + \frac{Ed_{31}(\alpha_e - 1)}{2D_E} \frac{K_E}{\sqrt{2\pi}}$		$-\frac{k_e(1-v)}{2D_D} \frac{K_{II}}{\sqrt{2\pi}}$
e	$-\frac{2(1+v)}{D_E} \frac{K_I}{\sqrt{2\pi}}$		$\frac{-k_e(1-v)(d_{33} - d_{31}) - 4d_{31}(1+v)}{2d_{31}D_D} \frac{K_I}{\sqrt{2\pi}} + \frac{k_e(1-v)}{2d_{31}D_D} \frac{K_D}{\sqrt{2\pi}}$
f	$\frac{2(1+v)}{D_E} \frac{K_{II}}{\sqrt{2\pi}} + \frac{Ed_{31}}{D_E} \frac{K_E}{\sqrt{2\pi}}$		$\frac{2(1+v)}{D_D} \frac{K_{II}}{\sqrt{2\pi}}$

The crack plane is perpendicular to poling direction. The potentials are $F'' = (a + ib)z^{-1/2}$, $G' = (c + id)z^{-1/2}$ and $H'' = (e + if)z_e^{-1/2}$. The field quantities can then be derived through Eqs. (4.11)–(4.18).

$$\frac{4(1+v)}{Ed_{31}}d + k_e\alpha_e \frac{1-v}{Ed_{31}}f = \frac{K_E}{\sqrt{2\pi}} \quad (4.24)$$

Eqs. (4.18)–(4.23) can be solved for the six real coefficients. These coefficients are listed on the left column of Table 1.

4.1.2. $\beta = 0$ crack perpendicular to poling direction, electrically impermeable

For the impermeable electrical conditions Eqs. (4.19)–(4.22) remain valid, however Eqs. (4.23) and (4.24) do not apply. The electrically impermeable crack boundary conditions, Eqs. (4.6) and (4.7), imply that

$$(d_{33} - d_{31})\left(b + \frac{3}{2}d\right) + 8\frac{1+v}{1-v} \frac{d_{31}}{k_e}d + (d_{33} + d_{31})f = 0 \quad (4.25)$$

$$(d_{33} - d_{31})\left(a + \frac{3}{2}c\right) + 8\frac{1+v}{1-v} \frac{d_{31}}{k_e}c + (d_{33} + d_{31})e = \frac{K_D}{\sqrt{2\pi}}. \quad (4.26)$$

Now, Eqs. (4.19)–(4.22), (4.25) and (4.26) are solved for the coefficients a – f . These coefficients are listed in the right column of Table 1.

Using the coefficients listed in Table 1 for the complex potentials of Eqs. (4.8)–(4.10), the stress, electric field, displacement and electric potential components can be determined from Eqs. (4.11)–(4.18). Strain and electric displacement components can be obtained from the constitutive law, i.e. Eqs. (2.12)–(2.17).

4.1.3. $\beta = \pi/2$ crack parallel to poling direction, electrically conducting

For the cases where the crack is parallel to the poling direction $\beta = \pi/2$. Again for $\beta = \pi/2$ it is useful to rewrite Eqs. (3.19)–(3.25). In this case, $p_e = i/\alpha_e = i\sqrt{1 - k_e}$ and $z_e = x + iy/\alpha_e$. Then, the stress, electric field, displacement and electric potential components are

$$\sigma_{xx} = -\operatorname{Re} F'' + \operatorname{Re} (2G' - \bar{z}G'') - \frac{1}{\alpha_e^2} \operatorname{Re} H'' \quad (4.27)$$

$$\sigma_{yy} = \operatorname{Re} F'' + \operatorname{Re} (2G' + \bar{z}G'') + \operatorname{Re} H'' \quad (4.28)$$

$$\sigma_{xy} = \operatorname{Im} F'' + \operatorname{Im} (\bar{z}G'') + \frac{1}{\alpha_e} \operatorname{Im} H'' \quad (4.29)$$

$$E_x = \frac{4(1+v)}{Ed_{31}} \operatorname{Re} G' - \frac{1-v}{Ed_{31}} k_e \operatorname{Re} H'' \quad (4.30)$$

$$E_y = -\frac{4(1+v)}{Ed_{31}} \operatorname{Im} G' + \frac{1-v}{Ed_{31}} \frac{k_e}{\alpha_e} \operatorname{Im} H'' \quad (4.31)$$

$$\frac{E}{1+v} u_x = -\operatorname{Re} F' + \operatorname{Re} \left[\left(7 + 4 \frac{d_{33} - d_{31}}{d_{31}} \right) G - \bar{z}G' \right] - \left(1 + \frac{1-v}{1+v} \frac{d_{33} - d_{31}}{d_{31}} \right) \operatorname{Re} H' \quad (4.32)$$

$$\frac{E}{1+v} u_y = \operatorname{Im} F' + \operatorname{Im} (7G + \bar{z}G') + \frac{1}{\alpha_e} \operatorname{Im} H' \quad (4.33)$$

$$\phi = -\frac{4(1+v)}{Ed_{31}} \operatorname{Re} G - \frac{1-v}{Ed_{31}} k_e \alpha_e^2 \operatorname{Re} H' \quad (4.34)$$

For the material poled parallel to the crack plane, under electrically conducting crack boundary conditions, Eqs. (4.1)–(4.5) imply that the coefficients in Eqs. (4.8)–(4.10) satisfy the following equations.

$$b + \frac{3}{2}d + f = 0 \quad (4.35)$$

$$-a + \frac{1}{2}c - \frac{1}{\alpha_e}e = 0 \quad (4.36)$$

$$a + \frac{3}{2}c + e = \frac{K_I}{\sqrt{2\pi}} \quad (4.37)$$

$$b - \frac{1}{2}d + \frac{1}{\alpha_e}f = \frac{K_{II}}{\sqrt{2\pi}} \quad (4.38)$$

$$4(1+v)d - k_e(1-v)f = 0 \quad (4.39)$$

$$\frac{4(1+v)}{Ed_{31}}c - k_e \frac{1-v}{Ed_{31}}e = \frac{K_E}{\sqrt{2\pi}} \quad (4.40)$$

Again, these six equations can be solved for the coefficients and the solutions are tabulated in the left column of Table 2.

4.1.4. $\beta = \pi/2$ crack parallel to poling direction, electrically impermeable

For the electrically impermeable conditions, Eqs. (4.39) and (4.40) are replaced by those corresponding to (4.6) and (4.7), which, for $\beta = \pi/2$, are

$$-(d_{33} - d_{31})a + \left(\frac{d_{33} - d_{31}}{2} + 8 \frac{1+v}{1-v} \frac{d_{31}}{k_e} \right) c - \frac{d_{33} + d_{31}}{\alpha_e} e = 0 \quad (4.41)$$

$$(d_{33} - d_{31})b - \left(\frac{d_{33} - d_{31}}{2} + 8 \frac{1+v}{1-v} \frac{d_{31}}{k_e} \right) d + \frac{d_{33} + d_{31}}{\alpha_e} f = \frac{K_D}{\sqrt{2\pi}} \quad (4.42)$$

Now, (4.35)–(4.38) and (4.41) and (4.42) can be solved for the coefficients, and these results are given in the right column of Table 2.

Using the coefficients listed in Table 2 for the complex potentials of Eqs. (4.8)–(4.10), the stress, electric field and displacement components can be determined from Eqs. (4.27)–(4.34). Strain and electric displacement components can be obtained from the constitutive law, i.e. Eqs. (2.12)–(2.17).

Table 2
The coefficients for the plane strain complex potentials with $\beta = \pi/2$

Plane strain	$\beta = \frac{\pi}{2}$	$k_e = \frac{2Ed_{31}^2}{\kappa(1-v)}$	$\alpha_e = \sqrt{\frac{1}{1-k_e}}$
	$D_E = k_e \alpha_e (1-v) + 2(\alpha_e - 1)(1+v)$		$D_D = k_e (1-v) + 2(\alpha_e - 1)(1+v)$
Conducting		Impermeable	
a	$\frac{k_e \alpha_e (1-v) - 8(1+v)}{4D_E} \frac{K_I}{\sqrt{2\pi}} + \frac{Ed_{31}(\alpha_e + 3)}{4D_E} \frac{K_E}{\sqrt{2\pi}}$	$\frac{k_e (1-v) - 8(1+v)}{4D_D} \frac{K_I}{\sqrt{2\pi}}$	
b	$\frac{3k_e \alpha_e (1-v) + 8\alpha_e (1+v)}{4D_E} \frac{K_{II}}{\sqrt{2\pi}}$		$\frac{-Ed_{31}(\alpha_e + 3)}{4\kappa D_D} \frac{K_D}{\sqrt{2\pi}} + \frac{3\kappa k_e (1-v) + 8\kappa \alpha_e (1+v) + Ed_{31}(d_{33} - d_{31})(\alpha_e + 3)}{4\kappa D_D} \frac{K_{II}}{\sqrt{2\pi}}$
c	$\frac{k_e \alpha_e (1-v)}{2D_E} \frac{K_I}{\sqrt{2\pi}} + \frac{Ed_{31}(\alpha_e - 1)}{2D_E} \frac{K_E}{\sqrt{2\pi}}$	$\frac{k_e (1-v)}{2D_D} \frac{K_I}{\sqrt{2\pi}}$	
d	$\frac{-k_e \alpha_e (1-v)}{2D_E} \frac{K_{II}}{\sqrt{2\pi}}$		$\frac{-Ed_{31}(\alpha_e - 1)}{2\kappa D_D} \frac{K_D}{\sqrt{2\pi}} + \frac{-k_e \kappa (1-v) + Ed_{31}(d_{33} - d_{31})(\alpha_e - 1)}{2\kappa D_D} \frac{K_{II}}{\sqrt{2\pi}}$
e	$\frac{2\alpha_e (1+v)}{D_E} \frac{K_I}{\sqrt{2\pi}} - \frac{\alpha_e Ed_{31}}{D_E} \frac{K_E}{\sqrt{2\pi}}$	$\frac{2\alpha_e (1+v)}{D_D} \frac{K_I}{\sqrt{2\pi}}$	
f	$\frac{-2\alpha_e (1+v)}{D_E} \frac{K_{II}}{\sqrt{2\pi}}$		$\frac{-2\alpha_e \kappa (1+v) - \alpha_e Ed_{31}(d_{33} - d_{31})}{\kappa D_D} \frac{K_{II}}{\sqrt{2\pi}} + \frac{\alpha_e Ed_{31}}{\kappa D_D} \frac{K_D}{\sqrt{2\pi}}$

The crack plane is parallel to poling direction. The potentials are $F'' = (a + ib)z^{-1/2}$, $G' = (c + id)z^{-1/2}$ and $H'' = (e + if)z_e^{-1/2}$. The field quantities can then be derived through Eqs. (4.27)–(4.34).

4.2. Plane stress crack tip fields

Under plane stress conditions, the derivatives of the complex potentials P , Q and S can be written as

$$P'' = (m + in)z^{-1/2} \quad (4.43)$$

$$Q'' = (p + iq)z_\sigma^{-1/2} \quad (4.44)$$

$$S'' = (r + is)z^{-1/2} \quad (4.45)$$

In the following subsections the conditions (4.1)–(4.7) will be applied to determine the real coefficients m , n , p , q , r and s .

4.2.1. $\beta = 0$ crack perpendicular to poling direction, electrically conducting

For $\beta = 0$, $p_\sigma = i\alpha_\sigma = i/\sqrt{1 - k_\sigma}$, with $z_\sigma = x + i\alpha_\sigma y$, where α_σ is real. Then, the stresses, electric fields, displacements and electric potential are

$$\sigma_{xx} = -\operatorname{Re} P'' - \alpha_\sigma^2 \operatorname{Re} Q'' \quad (4.46)$$

$$\sigma_{yy} = \operatorname{Re} P'' + \operatorname{Re} Q'' \quad (4.47)$$

$$\sigma_{xy} = \operatorname{Im} P'' + \alpha_\sigma \operatorname{Im} Q'' \quad (4.48)$$

$$E_x = \operatorname{Im} S'' + \frac{d_{31}}{\kappa} \alpha_\sigma \operatorname{Im} Q'' \quad (4.49)$$

$$E_y = \operatorname{Re} S'' + \frac{d_{31}}{\kappa} \alpha_\sigma^2 \operatorname{Re} Q'' \quad (4.50)$$

$$\frac{E}{1 + \nu} u_x = -\operatorname{Re} P' - \operatorname{Re} Q' + \frac{Ed_{31}}{1 + \nu} \operatorname{Re} S' \quad (4.51)$$

$$\frac{E}{1 + \nu} u_y = \operatorname{Im} P' + \alpha_\sigma \left(1 + \frac{k_\sigma}{1 + \nu} \frac{d_{33} - d_{31}}{d_{31}} \right) \operatorname{Im} Q' + \frac{Ed_{33}}{1 + \nu} \operatorname{Im} S' \quad (4.52)$$

$$\phi = -\operatorname{Im} S' - \frac{d_{31}}{\kappa} \alpha_\sigma \operatorname{Im} Q' \quad (4.53)$$

For the material poled perpendicular to the crack plane, under electrically conducting crack boundary conditions, Eqs. (4.1)–(4.5) imply that the coefficients in Eqs. (4.8)–(4.10) satisfy the following equations.

$$n + q = 0 \quad (4.54)$$

$$m + \alpha_\sigma p = 0 \quad (4.55)$$

$$m + p = \frac{K_I}{\sqrt{2\pi}} \quad (4.56)$$

$$n + \alpha_\sigma q = \frac{K_{II}}{\sqrt{2\pi}} \quad (4.57)$$

$$r + \frac{d_{31}}{\kappa} \alpha_\sigma p = 0 \quad (4.58)$$

$$s + \frac{d_{31}}{\kappa} \alpha_\sigma q = \frac{K_E}{\sqrt{2\pi}} \quad (4.59)$$

Eqs. (4.54)–(4.59) can be solved for the six real coefficients. These coefficients are listed on the left column of Table 3.

4.2.2. $\beta = 0$ crack perpendicular to poling direction, electrically impermeable

For the impermeable electrical conditions Eqs. (4.54)–(4.57) remain valid, however Eqs. (4.58) and (4.59) do not apply. The electrically impermeable crack boundary conditions, Eqs. (4.6) and (4.7), imply that

$$(d_{33} - d_{31})n + d_{33}q + \kappa s = 0 \quad (4.60)$$

$$(d_{33} - d_{31})m + d_{33}p + \kappa r = \frac{K_D}{\sqrt{2\pi}}. \quad (4.61)$$

Now, Eqs. (4.54)–(4.57) and (4.60) and (4.61) are solved for the coefficients. These coefficients are listed in the right column of Table 3.

Using the coefficients listed in Table 3 for the complex potentials of Eqs. (4.8)–(4.10), the stress, electric field, displacement and electric potential components can be determined from Eqs. (4.46)–(4.53). Strain and electric displacement components can be obtained from the constitutive law, i.e. Eqs. (2.12)–(2.17).

Table 3

The coefficients for the plane stress complex potentials with $\beta = 0$

Plane stress	$\beta = 0$	$k_\sigma = \frac{Ed_{31}^2}{\kappa}$	$\alpha_\sigma = \sqrt{\frac{1}{1-k_\sigma}}$
	Conducting		Impermeable
m	$\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$		$\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$
n	$-\frac{1}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$		$-\frac{1}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$
p	$-\frac{1}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$		$-\frac{1}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$
q	$\frac{1}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$		$\frac{1}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$
r	$\frac{d_{31}}{\kappa} \frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$		$-\left(\frac{d_{33}}{\kappa} - \frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{d_{31}}{\kappa} \right) \frac{K_I}{\sqrt{2\pi}} + \frac{K_D}{\kappa \sqrt{2\pi}}$
s	$-\frac{d_{31}}{\kappa} \frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}} + \frac{K_E}{\sqrt{2\pi}}$		$-\frac{d_{31}}{\kappa} \frac{1}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$

The crack plane is perpendicular to poling direction. The potentials are $P'' = (m + in)z^{-1/2}$, $Q'' = (p + iq)z_\sigma^{-1/2}$ and $S'' = (r + is)z^{-1/2}$. The field quantities can then be derived through Eqs. (4.46)–(4.53).

4.2.3. $\beta = \pi/2$ crack parallel to poling direction, electrically conducting

For the cases where the crack is parallel to the poling direction $\beta = \pi/2$. Again for $\beta = \pi/2$ it is useful to rewrite Eqs. (3.31)–(3.37). In this case, $p_\sigma = i/\alpha_\sigma = i\sqrt{1 - k_\sigma}$ and $z_\sigma = x + iy/\alpha_\sigma$. Then, the stress, electric field, displacement and electric potential components are

$$\sigma_{xx} = -\operatorname{Re} P'' - \frac{1}{\alpha_\sigma^2} \operatorname{Re} Q'' \quad (4.62)$$

$$\sigma_{yy} = \operatorname{Re} P'' + \operatorname{Re} Q'' \quad (4.63)$$

$$\sigma_{xy} = \operatorname{Im} P'' + \frac{1}{\alpha_\sigma} \operatorname{Im} Q'' \quad (4.64)$$

$$E_x = \operatorname{Im} S'' - \frac{d_{31}}{\kappa} \operatorname{Re} Q'' \quad (4.65)$$

$$E_y = \operatorname{Re} S'' + \frac{d_{31}}{\kappa} \frac{1}{\alpha_\sigma} \operatorname{Im} Q'' \quad (4.66)$$

$$\frac{E}{1+\nu} u_x = -\operatorname{Re} P' - \left(1 + \frac{k_\sigma}{1+\nu} \frac{d_{33} - d_{31}}{d_{31}}\right) \operatorname{Re} Q' + \frac{Ed_{33}}{1+\nu} \operatorname{Im} S' \quad (4.67)$$

$$\frac{E}{1+\nu} u_y = \operatorname{Im} P' + \frac{1}{\alpha_\sigma} \operatorname{Im} Q' - \frac{Ed_{31}}{1+\nu} \operatorname{Re} S' \quad (4.68)$$

$$\phi = -\operatorname{Im} S' + \frac{d_{31}}{\kappa} \operatorname{Re} Q' \quad (4.69)$$

For the material poled parallel to the crack plane, under electrically conducting crack boundary conditions, Eqs. (4.1)–(4.5) imply that the coefficients in Eqs. (4.8)–(4.10) satisfy the following equations.

$$n + q = 0 \quad (4.70)$$

$$m + \frac{1}{\alpha_\sigma} p = 0 \quad (4.71)$$

$$m + p = \frac{K_I}{\sqrt{2\pi}} \quad (4.72)$$

$$n + \frac{1}{\alpha_\sigma} q = \frac{K_{II}}{\sqrt{2\pi}} \quad (4.73)$$

$$\frac{d_{31}}{\kappa} q + r = 0 \quad (4.74)$$

$$s - \frac{d_{31}}{\kappa} p = \frac{K_E}{\sqrt{2\pi}} \quad (4.75)$$

Again, these six equations can be solved for the coefficients and the solutions are tabulated in the left column of Table 4.

Table 4

The coefficients for the plane stress complex potentials with $\beta = \pi/2$

Plane stress $\beta = \frac{\pi}{2}$ $k_\sigma = \frac{Ed_{31}^2}{\kappa}$ $\alpha_\sigma = \sqrt{\frac{1}{1-k_\sigma}}$			
	Conducting	Impermeable	
m	$-\frac{1}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$		$-\frac{1}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$
n	$\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$		$\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$
p	$\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$		$\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$
q	$-\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$		$-\frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$
r	$\frac{d_{31}}{\kappa} \frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_{II}}{\sqrt{2\pi}}$		$-\left(\frac{d_{33}}{\kappa} - \frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{d_{31}}{\kappa} \right) \frac{K_{II}}{\sqrt{2\pi}} + \frac{K_D}{\kappa \sqrt{2\pi}}$
s	$\frac{d_{31}}{\kappa} \frac{\alpha_\sigma}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}} + \frac{K_E}{\sqrt{2\pi}}$		$\frac{d_{31}}{\kappa} \frac{1}{\alpha_\sigma - 1} \frac{K_I}{\sqrt{2\pi}}$

The crack plane is parallel to poling direction. The potentials are $P'' = (m + in)z^{-1/2}$, $Q'' = (p + iq)z_\sigma^{-1/2}$ and $S'' = (r + is)z^{-1/2}$. The field quantities can then be derived through Eqs. (4.62)–(4.69).

4.2.4. $\beta = \pi/2$ crack parallel to poling direction, electrically impermeable

For the electrically impermeable conditions, Eqs. (4.74) and (4.75) are replaced by those corresponding to (4.6) and (4.7), which, for $\beta = \pi/2$, are

$$(d_{33} - d_{31})m + \frac{d_{33}}{\alpha_\sigma} p - \kappa s = 0 \quad (4.76)$$

$$(d_{33} - d_{31})n + \frac{d_{33}}{\alpha_\sigma} q + \kappa r = \frac{K_D}{\sqrt{2\pi}} \quad (4.77)$$

Now, (4.70)–(4.73) and (4.76) and (4.77) can be solved for the coefficients, and these results are given in the right column of Table 4.

Using the coefficients listed in Table 4 for the complex potentials of Eqs. (4.8)–(4.10), the stress, electric field, displacement and electric potential components can be determined from Eqs. (4.62)–(4.69). Strain and electric displacement components can be obtained from the constitutive law, i.e. Eqs. (2.12)–(2.17).

Note that in Tables 3 and 4 the coefficients m , n , p and q do not depend on the type of electrical boundary conditions specified in the problem. This result is due to the fact that the stresses in the plane stress problems only depend on two of the three potentials, and that the mechanical boundary conditions governing the asymptotic fields, Eq. (4.1), only specify tractions on the boundary. However, if the macroscopic outer problem contains displacement boundary conditions, then the stress intensity factors K_I and K_{II} can depend on the applied displacements and applied electrical loads.

4.3. Irwin matrices

The intensity factors K_I , K_{II} , and K_E or K_D characterize the mechanical and electrical fields in the vicinity of the crack tip, and are dependent on both specimen geometry and loading. As an example, consider a through-crack of length $2a$ lying along the x -axis in an infinite piezoelectric body subjected to far field stresses and electric fields σ_{xx}^∞ , σ_{xy}^∞ , σ_{yy}^∞ , E_x^∞ and E_y^∞ . The stress, electric field and electric displacement intensity factors for this type of specimen are, $K_I = \sigma_{yy}^\infty \sqrt{\pi a}$, $K_{II} = \sigma_{xy}^\infty \sqrt{\pi a}$, and $K_E = E_x^\infty \sqrt{\pi a}$ (conducting) or $K_D = D_y^\infty \sqrt{\pi a}$ (impermeable), where D_y^∞ is related to the far field stresses and electric fields through the appropriate constitutive equation, Suo et al. (1992). Note that these expressions are valid for any arbitrary value of the angle β , but in general, for other geometries or loadings, the expressions for the intensity factors will not be as simple as the ones listed above and will depend on β .

In addition to the intensity factors, another fracture quantity of interest is the energy release rate G . The energy release rate is directly related to the intensity factors. This relationship can be determined by performing a crack closure integral, i.e.

$$G\delta a = \frac{1}{2} \int_0^{\delta a} \sigma_{yy}(r) \Delta u_y(\delta a - r) + \sigma_{xy}(r) \Delta u_y(\delta a - r) + \phi(r) \Delta D_y(\delta a - r) dr \quad (\text{conducting}) \quad (4.78)$$

$$G\delta a = \frac{1}{2} \int_0^{\delta a} \sigma_{yy}(r) \Delta u_y(\delta a - r) + \sigma_{xy}(r) \Delta u_y(\delta a - r) + D_y(r) \Delta \phi(\delta a - r) dr \quad (\text{impermeable}) \quad (4.79)$$

Here, $f(r)$ represents the quantity ahead of the crack tip on the plane where $\theta = 0$, and $\Delta g(r) = g(r, \theta = \pi) - g(r, \theta = -\pi)$ represents the jump in the quantity behind the crack tip.

Equivalently, G can be evaluated with the electromechanical form of the J -integral as

$$G = J \equiv \int_{\Gamma} h n_x - \sigma_{ij} n_j u_{i,x} + D_i n_i E_x d\Gamma \quad (4.80)$$

where Γ is a counterclockwise contour (around a crack tip growing to the right) encircling the crack tip, and h is the electrical enthalpy, which for a linear piezoelectric material is given as

$$h = \frac{1}{2} (\sigma_{ij} \varepsilon_{ij} - E_i D_i) \quad (4.81)$$

The Irwin matrix, \mathbf{H} , relates the intensity factors, K_I , K_{II} , and K_E or K_D , to the energy release rate G . The relationship is given here as

$$G = (K_{II} \quad K_I \quad K_E) \begin{bmatrix} H_{11}^E & H_{12}^E & H_{13}^E \\ H_{12}^E & H_{22}^E & H_{23}^E \\ H_{13}^E & H_{23}^E & H_{33}^E \end{bmatrix} \begin{pmatrix} K_{II} \\ K_I \\ K_E \end{pmatrix} \quad (4.82)$$

for conducting crack boundary conditions, or

$$G = (K_{II} \quad K_I \quad K_D) \begin{bmatrix} H_{11}^D & H_{12}^D & H_{13}^D \\ H_{12}^D & H_{22}^D & H_{23}^D \\ H_{13}^D & H_{23}^D & H_{33}^D \end{bmatrix} \begin{pmatrix} K_{II} \\ K_I \\ K_D \end{pmatrix} \quad (4.83)$$

for impermeable crack boundary conditions. Note that the Irwin matrix is symmetric. If the Irwin matrix is known for any arbitrary angle β , then its components can readily be computed for any other angle, Suo et al. (1992). For example, take the unprimed components to be those when $\beta = 0$, then the components for some other angle β are

$$H'_{11} = H_{11} \cos^2 \beta + H_{22} \sin^2 \beta + 2H_{12} \sin \beta \cos \beta \quad (4.84)$$

Table 5

The Irwin matrices for plane stress (superscript σ), plane strain (superscript ε), conducting (superscript E) and impermeable (superscript D) crack boundary conditions

$$\begin{aligned}
 & k_\sigma = \sqrt{\frac{1}{1-k_\sigma}} \quad \left. \begin{aligned} k_\sigma \alpha_\sigma &= \frac{E d_{31}^2}{\kappa} \frac{1}{E} \\ \alpha_\sigma &= \sqrt{\frac{1}{1-k_\sigma}} \end{aligned} \right\} \rightarrow \mathbf{H}^{E,\sigma} = \begin{bmatrix} \frac{k_\sigma \alpha_\sigma}{2(\alpha_\sigma - 1)} \frac{1}{E} & 0 & -\frac{d_{31}}{2} \\ 0 & \frac{k_\sigma \alpha_\sigma}{2(\alpha_\sigma - 1)} \frac{1}{E} & 0 \\ -\frac{d_{31}}{2} & 0 & \frac{\kappa}{2} \end{bmatrix} \quad \mathbf{H}^{D,\sigma} = \begin{bmatrix} \frac{k_\sigma}{2(\alpha_\sigma - 1)} \frac{1}{E} & 0 & 0 \\ 0 & \frac{k_\sigma}{2} \left[\frac{\alpha_\sigma}{\alpha_\sigma - 1} - \left(\frac{d_{33}}{d_{31}} \right)^2 \right] \frac{1}{E} & \frac{d_{33}}{2\kappa} \\ 0 & \frac{d_{33}}{2\kappa} & -\frac{1}{2\kappa} \end{bmatrix} \\
 & D_E = k_\varepsilon \alpha_\varepsilon (1-v) + 2(\alpha_\varepsilon - 1)(1+v) \quad \left. \begin{aligned} D_D &= k_\varepsilon (1-v) + 2(\alpha_\varepsilon - 1)(1+v) \\ k_\varepsilon &= \frac{2E d_{31}^2}{\kappa(1-v)} \quad \alpha_\varepsilon = \sqrt{\frac{1}{1-k_\varepsilon}} \\ d_{15} &= d_{33} - d_{31} \end{aligned} \right\} \rightarrow \mathbf{H}^{E,\varepsilon} = \begin{bmatrix} \frac{2k_\varepsilon \alpha_\varepsilon}{D_E} \frac{1-v^2}{E} & 0 & -\frac{2(\alpha_\varepsilon - 1)(1+v)}{D_E} d_{31} \\ 0 & \frac{2k_\varepsilon \alpha_\varepsilon}{D_E} \frac{1-v^2}{E} & 0 \\ -\frac{2(\alpha_\varepsilon - 1)(1+v)}{D_E} d_{31} & 0 & \frac{D_D}{D_E} \frac{\kappa}{2} \end{bmatrix} \\
 & \mathbf{H}^{D,\varepsilon} = \begin{bmatrix} \frac{2k_\varepsilon}{D_D} \frac{1-v^2}{E} & 0 & 0 \\ 0 & \frac{-k_\varepsilon}{4D_D} \left\{ \left(\frac{d_{15}}{d_{31}} \right)^2 \frac{D_E}{1+v} + 8 \left[\alpha_\varepsilon \left(\frac{d_{15}}{d_{31}} \right) - \frac{d_{33}}{d_{31}} \right] \right\} \frac{1-v^2}{E} & \frac{1}{2D_D} \left[\left(\frac{d_{15}}{d_{31}} \right) D_E + 4(\alpha_\varepsilon - 1)(1+v) \right] \frac{d_{31}}{\kappa} \\ 0 & \frac{1}{2D_D} \left[\left(\frac{d_{15}}{d_{31}} \right) D_E + 4(\alpha_\varepsilon - 1)(1+v) \right] \frac{d_{31}}{\kappa} & -\frac{D_E}{D_D} \frac{1}{2\kappa} \end{bmatrix}
 \end{aligned}$$

All of these matrices are given for $\beta = 0$. Results for general β can be obtained through the use of Eqs. (4.84)–(4.89).

$$H'_{22} = H_{11} \sin^2 \beta + H_{22} \cos^2 \beta - 2H_{12} \sin \beta \cos \beta \quad (4.85)$$

$$H'_{12} = (H_{22} - H_{11}) \sin \beta \cos \beta + H_{12} (\cos^2 \beta - \sin^2 \beta) \quad (4.86)$$

$$H'_{13} = H_{13} \cos \beta + H_{23} \sin \beta \quad (4.87)$$

$$H'_{23} = -H_{13} \sin \beta + H_{23} \cos \beta \quad (4.88)$$

$$H'_{33} = H_{33} \quad (4.89)$$

where the \mathbf{H}' components are those for an arbitrary angle β as shown in Fig. 2. The unprimed components are given in Table 5 for plane strain, plane stress, conducting and impermeable crack boundary conditions.

5. Discussion

Complex potentials for the solution of in-plane, linear piezoelectric boundary value problems for a special class of materials with degenerate piezoelectric properties have been presented. This class of linear material properties is of considerable interest for non-linear constitutive models of ferroelectric behavior. It is envisioned that the asymptotic solutions presented here will be used to provide boundary conditions for “small scale switching” type analyses of the non-linear switching behavior near crack tips in ferroelectrics. Furthermore, when using the simplified set of constitutive properties, E , v , κ , d_{33} , d_{31} and $d_{15} = d_{33} - d_{31}$, the complex potentials can provide closed-form solutions to a wide range of boundary value problems. Such closed-form solutions are useful when attempting to ascertain the effects of the material properties on the electromechanical fields or other physical quantities in a given problem.

Finally, the complex potentials derived in Section 3 were applied to determine the asymptotic fields near a crack tip in a piezoelectric material. Solutions for both conducting and impermeable electrical crack face boundary conditions were obtained. The configurations with the crack perpendicular and parallel to the poling direction were solved explicitly, and the Irwin matrices were given in closed form for the crack plane oriented at any arbitrary angle to the poling direction.

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